ENTAILMENT AS ANALYTIC CONTAINMENT: TWO ALTERNATIVES TO E.

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Entailment is sometimes connected with synonymity in the following way. It is said that A and B are synonymous if and only if they entail one another. Again, one sense which has been given to entailment is the following: If A entails B, then the meaning of B is contained in the meaning of B. If synonymity is mutual entailment, then it would follow that A and B are synonymous if and only if they contain each others' meanings; i.e., if A and B contain all and only each others' meanings, then they have the same meaning, i.e., they are synonymous. Anderson and Belnap have pointed out a nice connection between (a) the notion of one expression's containing (or sharing) at least part of the meaning of another expression, and (b) the syntactical property of variable-sharing among the schemata which express the logical structures of the two expressions. Since, they hold, "A entails B", is true if and only if there is a valid inference from A to B, and since there cannot be a valid inference (on their view) unless A is relevant to B, and since A cannot be relevant to B in a given inference unless it is used in the inference to B, it turns out that every logical law of the form '(A entails B)' must have at least one variable in the formula A that occurs in the formula B.

In dealing with relevance, Anderson and Belnap rely more heavily on a device of subscripting steps in Fitch-style natural deducation rules, than they do on the concept of variable-sharing. Fully committed to Fitch-style natural deduction, they maintain that necessary and sufficient conditions of relevance can be explicated within that system by means of subscripting rules. Variable-sharing plays an important, but lesser, role. It is a necessary, but not a sufficient, condition of relevance. Everyone can agree with this; for A and -A will share all and only the same variables, but surely A will not in general entail -A. But the relative weakness of their use of variable-sharing comes out more clearly in the condition that at least one variable must be shared. In the following discussion I wish to discuss two systems of entailment, both independent of Anderson and Belnap's E and R, which satisfy the following stronger requirement:

> (1) If A entails B, then B contains only variables which occur in A

This rule, unlike Anderson and Belnap, does not allow the consequent B to contain any variables which do not occur in the antecedent. And it follows from (1) that

> (2) If A is synonymous with B (i.e., if A and B mutually entail each other) then B contains all and only those variables which occur in A and A contains all and only those variables which occur in B.

The philosophical intuition to which we appeal, in suggesting that (1) is a plausible requirement, is the Kantian notion that analytic truths are simply truths in which the predicate (or consequent) contains what is already contained in the subject (or antecedent). Though this notion has been rejected by various logicians and philosophers, it has a kind of simplicity and clarity which warrant further examination. A related philosophical intuition pertaining to (2) is that if two expressions mean the same thing exactly (are synonymous) then neither should contain a meaning the other lacks, or lack a component meaning the other possesses. This intuition is not implausible either, and whether correct or not, the two systems we wish to examine are systems which satisfy these two intuitions.

It follows immediately from (1), that the laws of addition (A→(AvB)), (B→(AvB)) - Axioms E8 and E9 of Anderson and Belnap's system of Entailment - will not be entailments, for they contain variables in the consequent which are not contained in the antecedent. It is also clear that the principle of transitivity cannot be expressed in the forms '((A→B)→((B→C)→(A→C)))' or '((A→B)→((C→A)→(C→B)))' for the same reason. On the other hand, we shall certainly want to retain principles of simplification, i.e., E4. ((A·B)→A) and E5. ((A·B)→B), since these seem to express precisely the notion of an entailment as having its consequent contain some part of the antecedent which is a necessary condition of the antecedent's truth. But if we have simplification, and the usual definitions of disjunction, etc., yet we wish to eliminate addition, then we must give up the principle of transposition with respect to entailment, i.e., we must give up Anderson and Belnap's E13, ((A→-B)→(B→-A)). These changes raise the question of what would be left, if we invoke the stricter rule, (1), as a necessary condition of relevance and thus of entailment. As a matter of fact the picture is not as dismal as it might seem, and in fact has several advantages over the Anderson-Belnap system E.

If we hold fast to our philosophical intuitions on entailment mentioned above, of course, we have good philosophical justification for these eliminations. The philosophical justification seems even clearer when we consider mutual entailment. For in Anderson and Belnap's system (A↔((AvB)·A)) is a theorem[1], but it is doubtful that intelligent users

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[1] The proof: 1. ((AvB)·A)→A) by E5, 2. (A→A) (a theorem of E), 3. (A→(AvB)) by E8, 4. ((A→(AvB))·(A→A)) by 2, 3, Adjunction, 5. (((A→(AvB))·(A→A))→(A→((AvB)·A))) by E6, 6. (A→((AvB)·A)) by 4, 5, M·P·, 7. (((AvB)·A)↔A) by 1, 6, Adjunction, Df. (See Axioms for E on p. 5.)

of ordinary language would incline to say that 'Joe spat' and '(Joe spat and either Nixon is guilty or Joe spat)' mutually entail each other, or mean exactly the same thing.

The following matrices provide a model for a possible system of entailment and synonymity which satisfies condition (1):

<u>Designated values</u>: 1, 3 (i.e., odd numbers)

–	A
2	1
1	2
4	3
3	4

·			
1	2	3	4
2	2	4	4
3	4	3	4
4	4	4	4

v			
1	1	3	3
1	2	3	4
3	3	3	3
3	4	3	4

⊃			
1	2	3	4
1	1	3	3
3	4	3	4
3	3	3	3

→			
1	2	4	4
1	1	4	4
3	4	3	4
3	3	3	3

↔			
1	2	4	4
2	1	4	4
4	4	3	4
4	4	4	3

We assume, as in E and R, two rules of inference:

<u>Modus Ponens</u>: From A→B and A, to infer B.
<u>Adjunction</u>: From A and B to infer (A·B).

Without providing an axiom set, it is easily shown by familiar methods that any axiomatic system which uses as axioms only schemata which take only the designated values 1 or 3 in their truth-tables, will be a consistent system. For the matrix for '·' shows that (A·B) will have an odd value (1 or 3) if and only if both A and B do; thus the rule of adjunction preserves the property of taking only odd values. Again the matrix for '→' shows that if A and (A→B) take odd values in any row, then B must take an odd value also; thus the rule of Modus Ponens will preserve the property of taking only odd values. Hence, given any initial set of schemata which have only odd values in their truth-table, all schemata derived by Modus Ponens or Adjunction will have only odd values. (Further, uniform substitution in tautologous schemata cannot introduce even values into the result of substitution, since all possible values which might appear in the substituenda are already accounted for). And finally, since the negation table always changes odd numbers to even numbers, it is obvious that we can never have both a theorem A and a theorem -A in such a system.

It is also clear, by inspection of the matrices, that no formula of the form (A→B) which is tautologous by these matrices can ever have a variable in the consequent which does not occur in the antecedent. This is easily proved as follows: (1) note that in all matrices a compound has the value 1 or 2 only if all components have the value 1 or 2 but if either component of a compound has the value 3 or 4, then the compound will have a value 3 or 4; (2) now suppose that (A→B) is a formula such that B contains some variable which does not occur in A; then assign 1 to all variables which occur in A and in 3 or 4 to any variable which is in B but not in A, then the antecedent A will have the value 1 or 2, and the consequent B will have the value 3 or 4. But (1→3), (1→4), (2→3), (2→4), all take undesignated values. Therefore there can be no entailment, on this model, if the consequent contains a variable not contained in the antecedent. It follows that mutual entailment, or synonymity, on this model, must have all and only the same variables.

Without attempting to establish an axiom set which is complete with respect to the tautologies (schemata taking only odd values in their final truth-table) in this model, I shall list a variety of possible axiom schemata which are or are not tautologies on this model; this will enable us to compare the principle (1) with E and R more carefully.

First, we note that all but four of the axiom schemata of E are tautologies on this model and two have closely related tautologies.

	E1.	$(((A\rightarrow A)\rightarrow B)\rightarrow B)$
<u>Not</u>	E2.	$((A\rightarrow B)\rightarrow((B\rightarrow C)\rightarrow(A\rightarrow C)))$ [But $(((A\rightarrow B)\cdot(B\rightarrow C))\rightarrow(A\rightarrow C))$ is a tautology]
	E3.	$((A\rightarrow(A\rightarrow B))\rightarrow(A\rightarrow B))$
	E4.	$((A\cdot B)\rightarrow A)$
	E5.	$((A\cdot B)\rightarrow B)$
	E6.	$(((A\rightarrow B)\cdot(A\rightarrow C))\rightarrow(A\rightarrow(B\cdot C)))$
	E7.	$((NA\cdot NB)\rightarrow N(A\cdot B))$ [$NA=df((A\rightarrow A)\rightarrow A)$]
<u>Not</u>	E8.	$(A\rightarrow(A\vee B))$
<u>Not</u>	E9.	$(B\rightarrow(A\vee B))$
	E10.	$(((A\rightarrow C)\cdot(B\rightarrow C))\rightarrow((A\vee B)\rightarrow C))$
	E11.	$((A\cdot(B\vee C))\rightarrow((A\cdot B)\vee C))$
	E12.	$((A\rightarrow -A)\rightarrow -A)$
<u>Not</u>	E13.	$((A\rightarrow -B)\rightarrow(B\rightarrow -A))$ [But $((A\rightarrow B)\cdot -B)\rightarrow -A)$ is a tautology]
	E14.	$--A\rightarrow A$

It is easily shown that systems satisfied by this model contain standard truth-functional logic (as does E). For

$$(A\rightarrow(A\cdot A))$$
$$((A\rightarrow B)\rightarrow(A\supset B))$$

are both tautologies and give us with E4, $((A\cdot B)\rightarrow A)$, the first two axioms of Rosser's system.

$$(A\supset(A\cdot A))$$
$$((A\cdot B)\supset A).$$

But Rosser's third axiom schema,

$$((A\supset B)\supset(-(B\cdot C)\supset -(C\cdot A)))$$

is also tautologous, as is the following version of truth-functional assertion.

$$((A\cdot(A\supset B))\rightarrow B)$$

Thus using Adjunction, and Modus Ponens we can derive the standard rule of detachment, <u>Given A and (A⊃B), infer B</u>. From this is follows that our model satisfies systems which contain standard logic (as does Anderson and Belnap's E). But, one great advantage this system has over Anderson and Belnap, both from and intuitive and from formal point of view, is its inclusion of

$$((A\cdot(A\supset B))\rightarrow B)$$
$$((-A\cdot(A\vee B))\rightarrow B)$$
$$((A\cdot -(A\cdot B))\rightarrow -B)$$

none of which are found in E, and each of which, the last two particularly, seem to capture traditional and obvious principles of inference and entailment. Furthermore, several other very plausible principles of entailment which are not in E, but are related to those above, are tautologies on this model, e.g.,

$$((A \rightarrow (B \vee C)) \rightarrow ((A \cdot -B) \rightarrow C))$$

Since our principle, (1), concerning relevance and entailment, is stronger in its restrictions than Anderson and Belnap's, it follows that ~~none of~~ the schemata which Anderson and Belnap wish to exclude on the basis of non-variable sharing, e.g., $(A \rightarrow (B \vee -B))$, $(A \rightarrow (B \rightarrow B))$, $((B \cdot -B) \rightarrow A)$, and many other "paradoxes of strict implication" ~~cannot~~ cannot be theorems of any system satisfying the present matrices.

It should also be pointed out that although transposition of entailment is not a principle of this system, many related principles, e.g., the principles of entailment which warrant the validity of "denying the consequent"

$$(((A \rightarrow B) \cdot -B) \rightarrow -A)$$
$$(((A \rightarrow -B) \cdot B) \rightarrow -A)$$
$$(((-A \rightarrow B) \cdot -B) \rightarrow A)$$
$$(((-A \rightarrow -B) \cdot B) \rightarrow A)$$

are tautologies. It is a question, which I shall not go into now, to what extent transposition of entailment is either needed or justified as long as we can use the argument form of denying the consequent.

It is interesting that in many cases principle (1) and the intuition behind it, help us to see clearly and immediately why certain schemata cannot be axiom schemata for entailment. For example,

(1) $(A\to((A\to B)\to B)$ (Assertion) cannot be a theorem in our systems (as it is not in E); although $((A\cdot(A\to B))\to B)$ is tautologous, thus a possible theorem. The reason for this difference is quite clear; the first has variables in its consequent that are not in its antecedent and the second does not.

(2) $((A\to(B\to C))\to((A\to B)\to(A\to C))$ is tautologous by our matrices (it is a thesis of E also), but $((A\to B)\to((A\to(B\to C))\to(A\to C)))$ (which is a thesis of E) is not tautologous. The explanation: while the first has no variables in its consequent which are not in its antecedent the second one does.

(3) $((A\to B)\to((B\to C)\to(A\to C)))$ and $((A\to B)\to((C\to A)\to(C\to B)))$ are not tautologies in our systems, (though they are both theses of E) although $(((A\to B)\cdot(B\to C))\to(A\to C))$ and $(((A\to B)\cdot(C\to A))\to(C\to B))$ are both tautologies in our systems. The reason for the difference: the same as in (2).

(4) In E, Permutation $((A\to(B\to C))\to(B\to(A\to C)))$ is a thesis, but it fails to come out a tautology on our matrices and thus cannot be a thesis on our systems. The reason: if it were a thesis then we could proceed from Self-distribution I, $((A\to(B\to C))\to((A\to B)\to(A\to C)))$, to Self-Distribution II, $((A\to B)\to((A\to(B\to C))\to(A\to C)))$ - but we have seen, in (2) above, that the first of these satisifes our principle while the second does not; hence inclusion of Permutation would lead to violations of our principle.

(5) In our systems, as in E, Importation $((A\to(B\to C))\to((A\cdot B)\to C)$ is (or can be) a thesis, but Exportation $((A\cdot B)\to C)\to(A\to(B\to C)))$ cannot. The reason: Importation satisfies our principal and cannot lead to its violation, since if the antecedent is tautologous, then $(B\to C)$ can have no variables not in A, and thus C cannot have any variables which are not in $(A\cdot B)$; in contrast, Exportation would allow us to proceed in any number of cases from a tautologous entailment to a non-tautologous one. E.g., from $(((A\to B)\cdot(B\to C))\to(A\to C))$ to $((A\to B\to((B\to C)\to(A\to C))$ (cf (3) above), or from $((A\cdot(A\to B))\to B)$ to $(A\to((A\to B)\to B))$ (cf (1) above), or from $((-A\cdot(A\vee B))\to B)$ to $(-A\to((A\vee B)\to B))$

The intuitive simplicity with which the inclusion and exclusion of our theses can be explained, as compared with the relatively obscure and unintuitive justification for the related principles in E, argues in favor of the systems I advocate.

But for all we have said, it is not a system with a <u>characteristic</u> matrix set like that given above which I wish to defend ultimately. I believe such a system comes closer to a simple and very workable sense of entailment and synonymity than E does, but it is still not satisfactory.

It would not work as a characteristic matrix for the necessity part of E since it is too loose: the matrices above also establish (A→((A→A)→A), i.e., A→NA. If we wish to have a matrix set which will capture Anderson and Belnap principles of necessity, even within the restrictions already mentioned, we will have to look for new matrices, which will avoid A→NA. This can no doubt be done without serious loss of the tautologies mentioned. But that is not our present interest. In fact we wish to restrict entailment and synonymity even further than we have done so far, and will deal with necessity at another time.

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[At this point, I have to quit writing this paper, so I shall sketch briefly what I wish to include in the rest of the paper. I shall first list a few schemata which would be admissible as entailments or synonyms under the matrices used above, but which it seems to me one could quite properly question as counter intuitive (e.g., ((A·-A)→(A→A))). Then I shall present an account of synonymity which I have presented before [cf. abstracts, "A Formalistic Approach to Synonymity", (J. of Phil., 11/7/68, pp. 712-713), and "A Unique Normal Form for Synonyms in the Propositional Calculus", (JSL, v. 38 (1973), p. 350)]. I shall show that this account, coupled with a recursive definition for inconsistency and a derived definition for tautology, yields a complete standard propositional calculus. Further, I shall define '(A entails B)' as 'A is synonymous with (A·B)'. It will turn out that A entails B if B is a conjunct in the (synonymous) conjunctive normal form of A. A semantics will be provided - if I have time to present it - in which it will be shown that we can identify conjuncts in the conjunctive normal form of A as a set of elementary,

<u>necessary</u> truth-conditions of the formula A; correspondingly the disjuncts in the disjunctive normal form of a formula A, constitute the set of elementary sufficient truth-conditions of A. A entails B, it will follow, if and only if every necessary truth-condition of B is a necessary truth-condition of A.